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# Renormalization of the vacuum angle in quantum mechanics, Berry phase and continuous measurements

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## Abstract

The vacuum angle  $\theta$  renormalization is studied for a toy model of a quantum particle moving around a ring, threaded by a magnetic flux  $\theta$ . A novel type of renormalization group (RG) transformation is introduced by coupling the particle to an additional slow variable, which may also be viewed as a part of a device, which ‘measures’ the particle’s position with finite accuracy. Then the renormalized  $\theta$  appears as a magnetic flux in the effective action for the slow ‘pointer’ variable. This ‘measurement-induced’ renormalization is shown to have the same properties as the  $\theta$  renormalization due to instantons in quantum field theories and leads to the RG flow diagram, similar to that of the quantum Hall effect, with observable effective  $\theta$  vanishing in the limit of small coupling between the particle and the measuring device.

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## 1. Introduction

In quantum chromodynamics (QCD) and similar theories it is possible to add a CP violating term  $i\theta Q$  to the Euclidean action, where  $Q$  is the topological charge and  $\theta$  is an additional parameter of the theory (see, e.g. [1]). Long ago it was suggested that vacuum angle  $\theta$  becomes scale dependent (as any other running coupling constant) if properly defined renormalization group (RG) transformation is introduced [2, 3], and flows to zero (mod  $2\pi$ ) in the infrared limit (see also [4] for some recent works). This essentially non-perturbative renormalization (the  $\theta$ -term has no effect in perturbation theory) is due to instantons of small size. Similar instanton-induced renormalization was also proposed for the Abelian Chern–Simons term in (2+1)-dimensional theories [5], which was shown to have no perturbative corrections beyond one loop [6].

The  $\theta$  renormalization, if taken literally, could possibly provide a solution to the strong CP problem in QCD (i.e. why we do not observe CP violation due to the  $\theta$ -term, for recent reviews see [7]). However, the  $\theta$  vacuum was initially introduced in such a way that  $\theta$  labelled different superselection sectors of the theory (see, e.g. [9]), i.e. it more resembled some conserved quantum number than the usual coupling constant and probably was not expected to be renormalized. Moreover, non-perturbative estimates [8] (see also [7] and references therein) have shown that CP violating effects actually depend on the bare  $\theta$ , so that it is not clear what the  $\theta$  renormalization actually means in QCD and how it may be observed.

Perhaps the most known example where such renormalization has proved to be important is the quantum Hall effect (QHE). In this case, described by a matrix nonlinear  $\sigma$  model, the renormalized vacuum angle is in fact defined as the observable Hall conductivity, dependent on the sample's size or temperature (see e.g. [3, 10]). Quite recently it became clear that charging effects in a single electron box (a metallic island coupled to the outside circuit by a tunnel junction), also described by a topological term, are closely related to the  $\theta$  renormalization [11–13]. This last model is equivalent to ordinary quantum mechanics of a particle (with friction in general case) on a ring threaded by a magnetic flux  $\theta$ , which can serve as the simplest zero-dimensional toy model to study the  $\theta$  renormalization in more detail.

It is possible to introduce different RG transformations for a particle on a ring [12, 13], which lead to the  $\theta$  renormalization. This toy model shows how the renormalization of  $\theta$  may be attributed to the loss of information about the initial topological charge when a given field configuration is coarse grained. But, since in these approaches the renormalization actually manifests itself, as in QHE, in a temperature dependence of a certain observable, they are not very helpful in understanding what  $\theta$  renormalization actually means in quantum field theories at strictly zero temperature.

For this reason here we present a novel RG scheme, inspired by an analogy between RG and continuous measurements, valid at zero temperature as well, which could be possibly generalized to the higher dimensional field theories. We introduce a ‘measuring device’, represented by an additional slow variable (a second heavy particle on the ring) coupled to the particle's coordinate and then define a renormalization as a dependence of its effective parameters on the coupling strength. Now the renormalized  $\theta$  appears as an effective magnetic flux seen by the heavy particle (or Berry phase, related to its slow rotation, compare with [14]). This may be understood, in a sense, as renormalization due to a continuous measurement of the quantum particle's position with finite accuracy. The resulting RG flow diagram again has the typical QHE-like form with  $\theta$  going to zero (mod  $2\pi$ ) in the ‘infrared’ limit of small coupling between the two particles. Physically such a behaviour is related to the change in the effective charge of the heavy particle due to rotations of the fast one, which is quite similar to the instanton-induced  $\theta$  renormalization in QCD or sigma models.

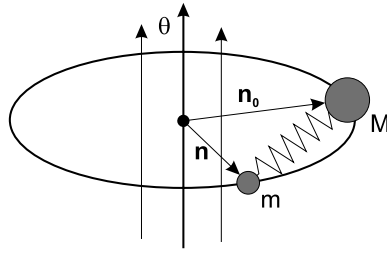
## 2. Renormalization for a particle on a ring

Consider a particle of mass  $m$  moving around a ring of unit radius threaded by a magnetic flux  $\theta$  (in units  $c = \hbar = e = 1$ ). The partition function for the model is given by the path integral

$$Z = \int D\mathbf{n}(\tau) \delta(\mathbf{n}^2(\tau) - 1) \exp(-S_0[\mathbf{n}]), \quad (1)$$

where  $\mathbf{n}(\tau)$  is the planar unit vector, which depends on the Euclidean time  $\tau \in [0, \beta]$  ( $\beta$  is the inverse temperature) and the integral is over periodic paths with  $\mathbf{n}(0) = \mathbf{n}(\beta)$ . The Euclidian action is

$$S_0[\mathbf{n}] = \frac{m}{2} \int_0^\beta \dot{\mathbf{n}}^2(\tau) d\tau - i \frac{\theta}{2\pi} \int_0^\beta \epsilon_{ab} n_a(\tau) \dot{n}_b(\tau) d\tau, \quad (2)$$



**Figure 1.** Two particles with different masses ( $M \gg m$ ) interacting via the harmonic potential on the ring with magnetic flux  $\theta$ .

where  $\epsilon_{ab}$  is the two-dimensional antisymmetric tensor. Since  $\mathbf{n}(0) = \mathbf{n}(\beta)$  the model is actually defined on a circle. The target space is also a circle ( $\mathbf{n}^2 = 1$ ), so that  $\mathbf{n}(\tau)$  is actually a mapping  $S^1 \rightarrow S^1$ . The last term in (2) has the form  $i\theta Q$  where  $Q$  is the topological charge which distinguishes inequivalent mappings and takes integer values (equal to a number of rotations the particle makes in time  $\beta$ ).

The model is in fact a (0+1)-dimensional quantum field theory with an action (2) similar, in a sense, to that of QCD with the  $\theta$ -term and also possesses instantons. Classical solutions with finite action obviously have the form  $\phi(\tau) = 2\pi k\tau/\beta$  ( $k$  is an integer) in terms of the polar angle  $\phi$  and correspond to  $k$  complete rotations in time  $\beta$ . In contrast to QCD action (2) is not scale invariant, so that there are no true instantons of arbitrary sizes, but there still exist small size instanton-like quantum fluctuations with nontrivial topological charge (fast rotations) that will be important for the renormalization to be introduced below.

The magnetic flux  $\theta$  explicitly breaks the  $T$  invariance, the most obvious  $T$ -violating effect being the non-zero persistent current in the ground state. This is the analogue of the CP problem in QCD and now one may ask how the dependence on  $\theta$  can be removed. One possible answer is that the magnetic flux could be screened if we allow the back reaction of the current on  $\theta$ . This may be done by introducing an additional dynamical variable (axion) coupled to the topological charge density. Curiously, model (2) with the axion has been introduced in a different context to describe a shunted Josephson junction [15]. If, however, no such back reaction is allowed, then all physical quantities (e.g. energy levels and correlation functions) obviously depend on the external flux  $\theta$ .

Suppose now that we cannot observe the particle's position  $\mathbf{n}(\tau)$  directly. Then we need some kind of a device to get an information about the system. One possible arrangement of the corresponding measuring apparatus is shown in figure 1. We introduce a second, almost classical particle with zero charge and a large mass  $M \gg m$ , which is coupled to the original one via the harmonic potential. Then  $\mathbf{n}_0$  may be viewed as a 'pointer' variable, 'measuring', in a sense, the quantum particle's position with an accuracy determined by the strength  $\lambda$  of the harmonic force (if e.g.  $\lambda \rightarrow \infty$  both particles are tightly bound and  $\mathbf{n}_0$  coincides with  $\mathbf{n}$ ). Or, to say this another way, the slow variable  $\mathbf{n}_0(\tau)$  may represent the coarse-grained trajectory of the fast quantum particle.

Below we will show that the effective magnetic flux, that is measured if only the heavy particle is available for observation is different from  $\theta$  and this change may be attributed to the renormalization due to instanton-like fluctuations of  $\mathbf{n}(\tau)$ , similar to that in the quantum field theory.

The Euclidean action for the whole system is

$$S[\mathbf{n}, \mathbf{n}_0] = S_0[\mathbf{n}] + \frac{M}{2} \int_0^\beta \dot{\mathbf{n}}_0^2(\tau) d\tau + \frac{\lambda}{2} \int_0^\beta [\mathbf{n}(\tau) - \mathbf{n}_0(\tau)]^2 d\tau. \quad (3)$$

If we now integrate over the fast variable  $\mathbf{n}(\tau)$ , then the partition function may be represented as

$$Z = \int D\mathbf{n}_0(\tau) \delta(\mathbf{n}_0^2(\tau) - 1) \exp\left(-\frac{M}{2} \int_0^\beta \dot{\mathbf{n}}_0^2(\tau) d\tau - S_{\text{eff}}[\mathbf{n}_0]\right), \quad (4)$$

where we introduce an effective action

$$\exp(-S_{\text{eff}}[\mathbf{n}_0]) = \int D\mathbf{n}(\tau) \delta(\mathbf{n}^2(\tau) - 1) w[\mathbf{n}, \mathbf{n}_0] \exp(-S_0[\mathbf{n}]), \quad (5)$$

and the weight functional  $w[\mathbf{n}, \mathbf{n}_0]$  has a simple Gaussian form

$$w[\mathbf{n}, \mathbf{n}_0] = \exp\left(-\frac{\lambda}{2} \int_0^\beta [\mathbf{n}(\tau) - \mathbf{n}_0(\tau)]^2 d\tau\right). \quad (6)$$

The main idea of the present paper is that the transformation (5), which defines the effective action for the slow variable in terms of the path integral over the fast one, may be viewed as a generalized Wilsonian RG transformation. From symmetry considerations, it is clear that when expanded in derivatives of  $\mathbf{n}_0$  the effective action should be of the same form as  $S_0$  but with new coupling constants, depending on the interaction strength  $\lambda$ . It is this dependence on  $\lambda$  that will be called ‘renormalization’ in what follows. In the real world  $\lambda$  of course has some fixed value so that the corresponding RG flow is difficult to observe, but if for some reasons  $\lambda$  is small then, as we will see below, the magnetic flux observed through measurements made on the heavy particle will also be small.

Before we proceed further it should be noted that equation (5) strongly resembles the so-called restricted path integral which appears in the theory of continuous quantum measurements. In the case of continuous monitoring of the observable  $\mathbf{n}(\tau)$  with  $\mathbf{n}_0(\tau)$  being the measurement record (this procedure is called a selective measurement), the particle’s propagator is given by a path integral similar to equation (5) and the weight functional  $w[\mathbf{n}, \mathbf{n}_0]$  is often taken in the Gaussian form, as in equation (6), where  $\lambda$  determines the accuracy of the measurement (see e.g. [16]). Hence one can say that the renormalization we study here is close to the selective continuous quantum measurement. It should be stressed however that here we deal with ‘measurement’ in Euclidean time, so that the above-mentioned analogy is not very close. Still we think that this is worth mentioning and may provide some deeper understanding of the transformation (5).

Now, a few comments are in order concerning the meaning of the transformation (5) as a renormalization. If we e.g. apply the same prescription to the 2D  $O(N)$   $\sigma$  model then in the one-loop calculation of [17],  $\lambda$  effectively acts as a mass squared for Goldstone modes with the charge renormalization  $\sim \ln(\Lambda/\sqrt{\lambda})$  ( $\Lambda$  is the ultraviolet cutoff). Hence changing  $\lambda$  is indeed similar to changing the scale. We now argue that beyond the perturbation theory,  $\lambda$  also may be viewed as a scale parameter.

For  $\lambda$  large enough only paths close to  $\mathbf{n}_0(\tau)$  contribute to the path integral (5). But for the particle on the ring it is possible that a given path  $\mathbf{n}(\tau)$  is close to  $\mathbf{n}_0(\tau)$  for most of the time, but suddenly makes a fast complete rotation around the ring in time  $\tau_0$ . For such instanton-like paths the weight factor (6) behaves as  $w \sim \exp(-\text{const} \times \lambda \tau_0)$ , so that large ‘instantons’ with size  $\tau_0 > 1/\lambda$  (slow rotations) are strongly suppressed (very fast rotations with  $\tau_0 \ll m$  are suppressed by the kinetic term in equation (2)). Then with decreasing  $\lambda$  more and more instanton-like paths of larger scale contribute to the integral (5). Clearly, this is exactly what a physicist usually expects from the RG transformation in theories with instantons.

### 3. Effective action and renormalized parameters

To obtain the effective action from equation (5) we first note that if  $\mathbf{n}_0(\tau)$  is fixed then the action for the fast particle (up to a constant, since  $\mathbf{n}^2 = \mathbf{n}_0^2 = 1$ ) may be written as

$$S[\mathbf{n}] = S_0[\mathbf{n}] - \lambda \int_0^\beta \mathbf{n}(\tau) \mathbf{n}_0(\tau) d\tau \quad (7)$$

and describes the particle on the ring in a time-dependent external electric field  $\lambda \mathbf{n}_0(\tau)$  (with  $\mathbf{n}_0(0) = \mathbf{n}_0(\beta)$ ). For slowly varying  $\mathbf{n}_0(\tau)$  at zero temperature one can treat this problem in the adiabatic approximation.

There are many ways to obtain the desired result but first we evaluate the magnetic flux that the heavy particle feels. This can be done e.g. by calculating the phase factor associated with the adiabatic cyclic evolution of  $\mathbf{n}_0(\tau)$ . If the fast particle was initially in its ground state in the presence of the electric field  $\lambda \mathbf{n}_0$  then, after adiabatic  $2\pi$ -rotation of the field, the ground state will turn back to itself up to a phase factor (Berry phase [18]) which we denote by  $\exp(i\theta')$ , with  $\theta'$  being the effective flux. If we introduce polar angles  $\phi$  and  $\phi_0$  instead of the vectors  $\mathbf{n}$  and  $\mathbf{n}_0$  then the corresponding Hamiltonian may be written as

$$H = \frac{1}{2m} \left( -i \frac{\partial}{\partial \phi} - \frac{\theta}{2\pi} \right)^2 + \lambda \cos(\phi - \phi_0(t)). \quad (8)$$

Let  $\psi_0(\phi) = \psi_0(\phi - \phi_0)$  be the instantaneous ground-state wavefunction for the Hamiltonian (8) with the energy  $E_0$ , which obviously does not depend on  $\phi_0$ . Then the Berry phase for the adiabatic change of  $\phi_0$  from zero to  $2\pi$  is given by the standard formula [18]

$$\theta' = i \int_0^{2\pi} d\phi_0 \langle \psi_0 | \frac{\partial}{\partial \phi_0} | \psi_0 \rangle. \quad (9)$$

Since  $\psi_0$  depends only on the difference  $\phi - \phi_0$  we have

$$\langle \psi_0 | \frac{\partial}{\partial \phi_0} | \psi_0 \rangle = - \langle \psi_0 | \frac{\partial}{\partial \phi} | \psi_0 \rangle = - \langle \psi_0 | \left( \frac{\partial}{\partial \phi} - i \frac{\theta}{2\pi} \right) | \psi_0 \rangle - i \frac{\theta}{2\pi} \quad (10)$$

The first term on the rhs of equation (10) is proportional to the average of the derivative  $\partial H / \partial \theta$  and hence

$$\theta' = \theta - 4\pi^2 m \frac{\partial E_0}{\partial \theta}. \quad (11)$$

The nontrivial Berry phase, different from  $\theta$ , means that the coarse-grained ('continuously measured') trajectory sees a 'renormalized' magnetic field, as was discussed in [12], due to unobservable fast instanton-like rotations. This implies that for slowly varying  $\mathbf{n}_0$  we should have (up to a constant)

$$S_{\text{eff}}[\mathbf{n}_0] = -i \frac{\theta'}{2\pi} \int_0^\beta \epsilon_{ab} n_0^a(\tau) \dot{n}_0^b(\tau) d\tau + \frac{m'}{2} \int_0^\beta \dot{\mathbf{n}}_0^2(\tau) d\tau + \dots, \quad (12)$$

where dots indicate terms with higher derivatives of  $\mathbf{n}_0$  and higher powers of  $\dot{\mathbf{n}}_0$  and the renormalized mass  $m'$  will be determined below.

A similar origin of topological terms from a corresponding Berry phase was discussed in detail in [14] where fermions were coupled to the background vector field in various space-time dimensions (fermionic  $\sigma$ -models). Then integration over fermions results in equation (12) for planar vector  $\mathbf{n}_0$  with  $\theta'$ ,  $m'$  dependent on the coupling constants. Here the fast mode which is integrated out is also the planar vector, so that it is more natural to speak of the  $\theta$  renormalization rather than of the induced topological term.

There exists a simple heuristic way to derive the expansion of equation (12). Consider a reference frame rotating with an angular frequency  $\omega = \dot{\phi}_0$ , which is assumed to be small and almost constant. In this frame  $\mathbf{n}_0$  is constant, but an additional magnetic field  $2m\omega$  is present according to Larmor's theorem. Hence the Hamiltonian  $H'$  in the rotating frame should be taken at the shifted value of the vacuum angle  $\theta + 2m\pi\omega$ , or more precisely,

$$H' = H + i\omega \frac{\partial}{\partial \phi} = H(\theta + 2m\pi\omega) - \frac{\theta}{2\pi}\omega - \frac{m}{2}\omega^2 \quad (13)$$

(see e.g. [19]), where the last term is the centrifugal potential (for the thin ring of unit radius) and the second one is due to the presence of the magnetic flux  $\theta$ . Then if the particle is in its ground state the effective action (after Wick rotation  $t \rightarrow -i\tau$  and expansion in powers of  $\dot{\phi}_0$ ) may be written as

$$\begin{aligned} S_{\text{eff}} &\simeq \int_0^\beta d\tau \left[ \frac{m}{2} \dot{\phi}_0^2 - i \frac{\theta}{2\pi} \dot{\phi}_0 + E_0(\theta + 2m\pi\dot{\phi}_0) \right] \\ &= \text{const} + \int_0^\beta d\tau \left[ \frac{m'}{2} \dot{\phi}_0^2 - i \frac{\theta'}{2\pi} \dot{\phi}_0 + \dots \right], \end{aligned} \quad (14)$$

where  $\theta'$  is given by the previously derived formula (11) and

$$m' = m - 4m^2\pi^2 \frac{\partial^2 E_0}{\partial \theta^2}. \quad (15)$$

Clearly, this is the same action as in equation (12). Formulae (11) and (15) formally look very similar to the RG equations derived in [12], though the RG transformation used here is quite different. Note that they are independent of the specific form of the coupling between  $\mathbf{n}$  and  $\mathbf{n}_0$ —all details are hidden in the ground-state energy  $E_0(\theta)$ . The dependence of  $E_0$  on  $\theta$  in a general case is rather well known by now, since it determines persistent currents in mesoscopic rings (see e.g. [20]).

#### 4. Exact renormalization group flow

For large  $\lambda$ , when the effective electric field is strong, the  $\theta$  dependence of  $E_0$  is suppressed and  $\theta' \simeq \theta$ . In this case  $E_0$  depends on  $\theta$  only through instantons, as discussed in detail in [21], and

$$E_0(\theta) \simeq \text{const} - 2\sqrt{S_0}K e^{-S_0} \cos \theta, \quad (16)$$

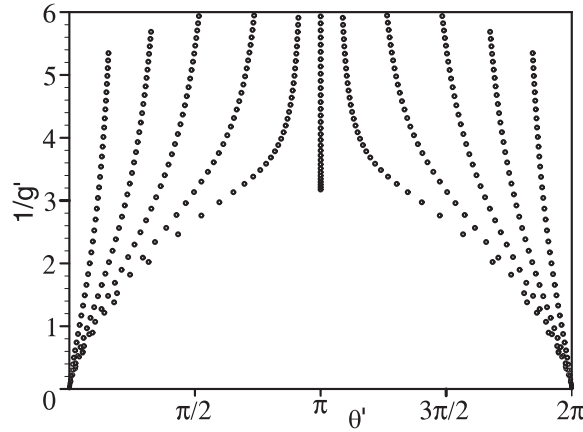
where  $S_0(\lambda) \sim \sqrt{m\lambda}$  is the classical instanton action and  $K = K(\lambda)$  results from the ratio of determinants [21]. Then in terms of dimensionless 'coupling constants'  $g = 1/\sqrt{m\lambda}$  and  $g' = 1/\sqrt{m'\lambda}$ , we finally have at  $g \rightarrow 0$

$$\theta' \simeq \theta - D(g) e^{-c/g} \sin \theta, \quad \frac{1}{g'^2} \simeq \frac{1}{g^2} - \frac{1}{g^2} D(g) e^{-c/g} \cos \theta, \quad (17)$$

where  $c$  is some numerical constant and  $D(g) = 8\pi^2 m K \sqrt{S_0}$ . These equations are qualitatively similar to  $\theta$  and charge renormalization due to instantons in QCD and  $\sigma$  models [2, 3].

If on the other hand  $\lambda$  tends to zero, then for the free motion on the ring  $E_0 = (1/2m)(\theta/2\pi)^2$  for  $\theta < \pi$ ,  $E_0 = (1/2m)(\theta/2\pi - 1)^2$  for  $\theta > \pi$  and equations (11), (15) imply that  $m' \rightarrow 0$  while  $\theta' \rightarrow 0$ ,  $\theta < \pi$  and  $\theta' \rightarrow 2\pi$ ,  $\theta > \pi$ . These results are almost obvious because at  $\lambda = 0$  the slow field  $\mathbf{n}_0$  is no longer coupled to  $\mathbf{n}$ .

In the close vicinity of the point  $\theta = \pi$  the situation is more complicated. At  $\lambda = 0$  the ground state is degenerate, but the degeneracy is lifted by an arbitrarily small external potential



**Figure 2.** Renormalized parameters  $1/g' = \sqrt{m'\lambda}$  and  $\theta'$  from equations (11), (15) for different values of initial  $\theta$ .  $\lambda$  decreases from top to bottom.

and an energy gap  $\delta E$  appears. At small  $\lambda$   $\delta E = a\sqrt{\lambda^2 + b(\theta - \pi)^2}$ , where  $a$  and  $b$  are some numerical constants, and after expanding in  $(\theta - \pi)$  near the maximum of  $E_0(\theta)$  at  $\theta = \pi$ , we have

$$E_0(\theta) \simeq \text{const} - \frac{\alpha}{2\lambda}(\theta - \pi)^2, \tag{18}$$

where  $\alpha = ab$ . Hence from equation (15)  $m' \rightarrow 4m^2\pi^2\alpha/\lambda$  at  $\lambda \rightarrow 0$  and

$$1/g' = \sqrt{m'\lambda} \rightarrow 2m\pi\sqrt{\alpha} = \text{const}, \quad \theta = \pi \tag{19}$$

Thus for  $\theta = \pi$  the coupling constant  $g'$  tends to a fixed value as  $\lambda \rightarrow 0$ . This is a kind of anomaly (similar to ‘rotational anomaly’ of [19]), since strictly at  $\lambda = 0$  there is no interaction and  $m'$  should be equal to zero. Certainly, for very small  $\lambda$  when  $\delta E$  tends to zero near  $\theta = \pi$  the adiabatic approximation used here becomes invalid.

Thus the dependence of  $m'$  and  $\theta'$  on  $\lambda$  reproduces the main features of the famous QHE RG flow diagram. This can be seen from figure 2, where the evolution of the renormalized parameters is shown with  $\lambda$  decreasing from top to bottom for different initial values of the vacuum angle  $\theta$ . The points in figure 2 result from the numerical calculation for a simplified model when the term  $\lambda \cos \phi$  in equation (8) is replaced with  $\lambda\delta(\phi)$  (qualitative features should not depend on the particular choice of the potential in equation (8)). Clearly, figure (2) is similar to the upper half of the QHE RG flow diagram with the unstable fixed point at  $\theta = \pi$  and the ultimate flow of the renormalized vacuum angle to zero (mod  $2\pi$ ).

The quantum mechanical model discussed here enables, however, a transparent explanation of why the effective  $\theta$  should vanish as  $\lambda \rightarrow 0$ . Let us return to figure 1 with two particles of masses  $m$  and  $M \gg m$  interacting via the harmonic potential. Note that initially only the light particle interacts with the magnetic flux  $\theta$ . One can say that the light particle is charged with, say, unit charge, while the heavy one is neutral.

Now if  $\lambda$ , which determines the interparticle interaction strength, is high enough, two particles form a tightly bound pair or an ‘atom’, exactly with unit total charge. Mathematically this means that the topological term for the field  $\mathbf{n}_0$  is induced with  $\theta' \simeq \theta$  due to the ‘condensation’ of charge near the point  $\mathbf{n}_0$ . When  $\lambda$  decreases, the bound state gets more loose. When the size of the bound state is of the order of the ring’s radius, rotations of the light particle are allowed (instanton-like fluctuations) and its charge is spread along the ring.



So the effective charge of the heavy particle reduces, which is seen in the formalism as the magnetic flux  $\theta$  renormalization.

## 5. Summary and discussion

In summary, we have demonstrated how the  $\theta$  renormalization may appear in the quantum mechanics of a particle, moving around a thin ring threaded by a magnetic flux  $\theta$ . To define the nonperturbative RG procedure we introduce an additional slow degree of freedom, which is the coordinate of a second heavy quasiclassical particle, but may also be viewed as a part of the measuring device ('pointer' observable) designed to measure the position of the initial particle with some finite accuracy.

Then RG equations describe the evolution of effective parameters of the heavy particle, when the coupling between the two particles is changed. For example, the renormalized  $\theta$  is the coefficient in front of the induced topological term in the effective action for slow variables (i.e. the magnetic flux that the heavy particle feels) after fast variables are integrated away. Probably, this is the most natural way in how the  $\theta$  renormalization may manifest itself in quantum field theories with instantons. This effective  $\theta$  is also equal to the Berry phase, associated with the cyclic evolution of the slow variables (see also [14]).

This is a somewhat unusual definition of RG transformation, since normally one expects that running couplings should depend on some momentum variable. However, this does not contradict the modern understanding of the Wilsonian approach to the exact RG (see e.g. [22]) and is, in a sense, similar to the so-called 'RG in the internal space' [23]. Moreover, it should be noted that the procedure used here is somewhat close in spirit to the way the running  $\theta$  appears in the Seiberg–Witten model [24].

The so constructed RG procedure leads precisely to the same  $\theta$  renormalization that was found in the weak coupling limit in different field theories (instanton-induced renormalization) and results in the typical RG flow diagram similar to that of the quantum Hall effect, which here can be calculated exactly.

The toy model studied here shows that while the renormalization of the vacuum angle is definitely a generic property of a system with instanton-like fluctuations it does not necessarily mean that low energy observables are independent of  $\theta$ , but is revealed, when the system is being 'measured', i.e. coupled in a special way to some additional slow variable. This mechanism, which may lead to small  $\theta$  in the effective theory, looks physically different from the direct screening of  $\theta$ , as e.g. in the case when the axion field is added. In more realistic truly nonlinear theories both fast and slow variables may be parts of the system itself, if e.g. the effective fields we observe are different from those directly coupled to  $\theta$ .

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## References

- [1] Polyakov A M 1987 *Gauge Fields and Strings* (New York: Harwood Academic)
- [2] Knizhnik V G and Morozov A Yu 1984 *JETP Lett.* **39** 240  
Levine H and Libby S 1985 *Phys. Lett. B* **150** 182

- [3] Levine H, Libbi S and Pruisken A M M 1984 *Nucl. Phys. B* **240** [FS12] 30  
Pruisken A M M 1987 *Nucl. Phys. B* **290** 61
- [4] Asorey M, Esteve J G and Salas J 1993 *Phys. Rev. B* **48** 3626  
Latorre J I and Lütken C A 1998 *Phys. Lett. B* **421** 217  
Pruisken A M M, Baranov M A and Voropaev M 2001 *Preprint cond-mat/0101003*  
Campos Venuti L, Degli Esposti Boschi C, Ercolessi E, Ortolani F, Morandi G, Pasini S and Roncaglia M 2005  
*J. Stat. Mech.* **L02004**  
Pruisken A M M, Shankar R and Surendran N 2005 *Phys. Rev. B* **72** 035329  
Pruisken A M M and Burmistrov I S 2005 *Ann. Phys., NY* **316** 285
- [5] Morozov A Yu and Niemi A J 1990 *Phys. Lett. B* **236** 44
- [6] Coleman S and Hill B 1985 *Phys. Lett. B* **159** 184  
Semenoff G W, Sodano P and Wu Y S 1989 *Phys. Rev. Lett.* **62** 715
- [7] Peccei R D 1998 Reflections on the strong CP problem *Preprint hep-ph/9807514*  
Srednicki M 2002 Axions: Past, Present, and Future *Preprint hep-th/0210172*  
Gabadadze G and Shifman M 2002 *Int. J. Mod. Phys. A* **17** 3689
- [8] Crewther R J, di Vecchia P, Veneziano G and Witten E 1979 *Phys. Lett. B* **88** 123  
Shifman M A, Vainshtein A I and Zakharov V I 1980 *Nucl. Phys. B* **166** 493
- [9] Callan C G, Dashen R and Gross D J 1978 *Phys. Rev. D* **17** 2717
- [10] Pruisken A M M 1990 *The Quantum Hall Effect* ed R E Prange and S Girvin (Berlin: Springer)  
Pruisken A M M and Burmistrov I S 2007 *Ann. Phys., NY* **322** 1265
- [11] Bulgadaev S A 2006 *JETP Lett.* **83** 659
- [12] Apenko S M 2006 *Phys. Rev. B* **74** 193311
- [13] Burmistrov I S and Pruisken A M M 2007 *Preprint cond-mat/0702400*
- [14] Stone M 1986 *Phys. Rev. D* **33** 1191  
Abanov A G and Wiegmann P B 2000 *Nucl. Phys. B* **570** 685
- [15] Apenko S M 1989 *Phys. Lett. A* **142** 277  
Schön G and Zaikin A D 1990 *Phys. Rep.* **198** 237
- [16] Mensky M B 1993 *Continuous Quantum Measurements and Path Integrals* (Bristol: Institute of Physics Publishing)  
Diósi L and Halliwell J J 1998 *Phys. Rev. Lett.* **81** 2846
- [17] Polyakov A M 1975 *Phys. Lett. B* **59** 79
- [18] Berry M 1984 *Proc. R. Soc. A* **392** 45
- [19] Merlin R 1993 *Phys. Lett. A* **18** 421
- [20] Cheung H F, Gefen Y, Riedel E K and Shih W H 1988 *Phys. Rev. B* **37** 6050
- [21] Rajaraman R 1982 *Solitons and Instantons* (Amsterdam: North Holland)
- [22] Morozov A and Niemi A J 2003 *Nucl. Phys. B* **666** 311
- [23] Polonyi J and Sailer K 2005 *Phys. Rev. D* **71** 025010
- [24] Seiberg N and Witten E 1994 *Nucl. Phys. B* **426** 19  
Konishi K 1997 *Phys. Lett. B* **392** 101